Vector Identity 3

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$$

Proof

Let $\mathbf{A}=\mathbf{a}$, $\mathbf{B}=\mathbf{b}$, $\mathbf{C}=\mathbf{c}$, and $\mathbf{D}=\mathbf{d}$.

$$\begin{aligned}
(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) &= \left[\left(\sum_{i=1}^{3}\boldsymbol{\delta}_i A_i\right)\times\left(\sum_{j=1}^{3}\boldsymbol{\delta}_j B_j\right)\right]\cdot\left[\left(\sum_{k=1}^{3}\boldsymbol{\delta}_k C_k\right)\times\left(\sum_{l=1}^{3}\boldsymbol{\delta}_l D_l\right)\right] \\
&= \left[\sum_{i=1}^{3}\sum_{j=1}^{3}(\boldsymbol{\delta}_i\times\boldsymbol{\delta}_j)A_iB_j\right]\cdot\left[\sum_{k=1}^{3}\sum_{l=1}^{3}(\boldsymbol{\delta}_k\times\boldsymbol{\delta}_l)C_kD_l\right] \\
&= \left(\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{m=1}^{3}\boldsymbol{\delta}_m\varepsilon_{mij}A_iB_j\right)\cdot\left(\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{n=1}^{3}\boldsymbol{\delta}_n\varepsilon_{nkl}C_kD_l\right) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3}\sum_{n=1}^{3}(\boldsymbol{\delta}_m\cdot\boldsymbol{\delta}_n)\varepsilon_{mij}\varepsilon_{nkl}A_iB_jC_kD_l \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3}\sum_{n=1}^{3}\delta_{mn}\varepsilon_{mij}\varepsilon_{nkl}A_iB_jC_kD_l \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3}\varepsilon_{mij}\varepsilon_{mkl}A_iB_jC_kD_l \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk})A_iB_jC_kD_l \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\delta_{ik}\delta_{jl}A_iB_jC_kD_l - \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\delta_{il}\delta_{jk}A_iB_jC_kD_l \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\delta_{ik}A_iB_jC_kD_j - \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{l=1}^{3}\delta_{il}A_iB_jC_jD_l \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}A_iB_jC_iD_j - \sum_{i=1}^{3}\sum_{j=1}^{3}A_iB_jC_jD_i \\
&= \left(\sum_{i=1}^{3}A_iC_i\right)\left(\sum_{j=1}^{3}B_jD_j\right) - \left(\sum_{i=1}^{3}A_iD_i\right)\left(\sum_{j=1}^{3}B_jC_j\right) \\
&= (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C})
\end{aligned}$$